Efficient Linear System Solver with Transformers

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Key Findings

- 🚀 Linear Transformers can efficiently solve small positive definite symmetric linear systems.
- 🔄 Effective reparametrization allows for solving problems with different lineary system sizes.
- 🏆 Competitive performance with classical methods for small systems.

Problem formulation

Find vector $x \in \mathbb{R}^N$ that solves the system of *N* linear equations: $\langle a_i, x \rangle = b_i$

With $a_i \in \mathbb{R}^N$ and $b_i \in \mathbb{R}$

Training data: positive definite symmetric matrices *A* with a fixed condition number.

Linear Transformer

Linear Transformer updates each layer using

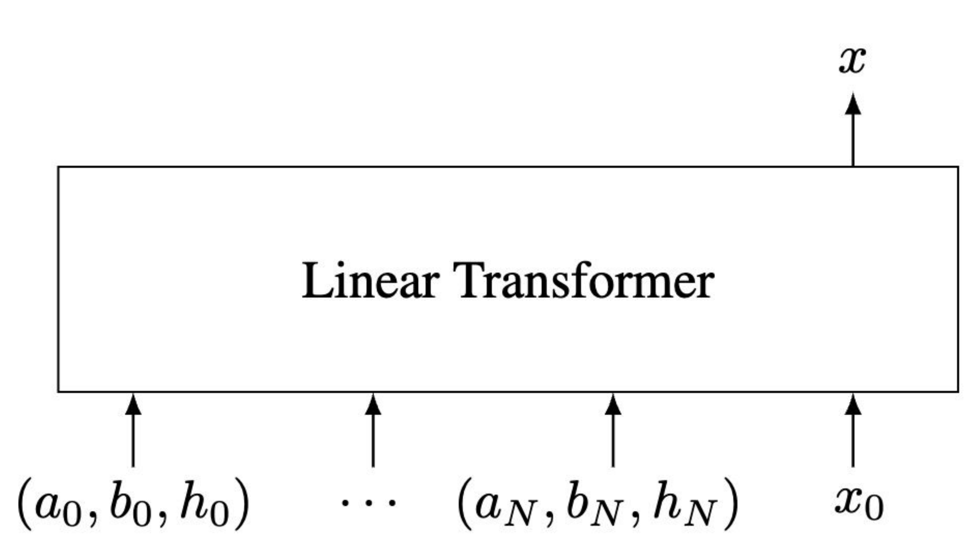
$$\Delta e_i = \sum_{j=1}^{N} (e_j^\top Q e_i) P e_j .$$

With weights $P = W_P W_V$ and $Q = W_K^\top W_Q$.

Objective function to minimize:

$$L(\theta) = \mathop{\mathbb{E}}_{A,b} \left[(f_\theta(\{e_1, \ldots, e_N\}, e_{N+1}) - x)^2 \right] .$$

Data Encoding

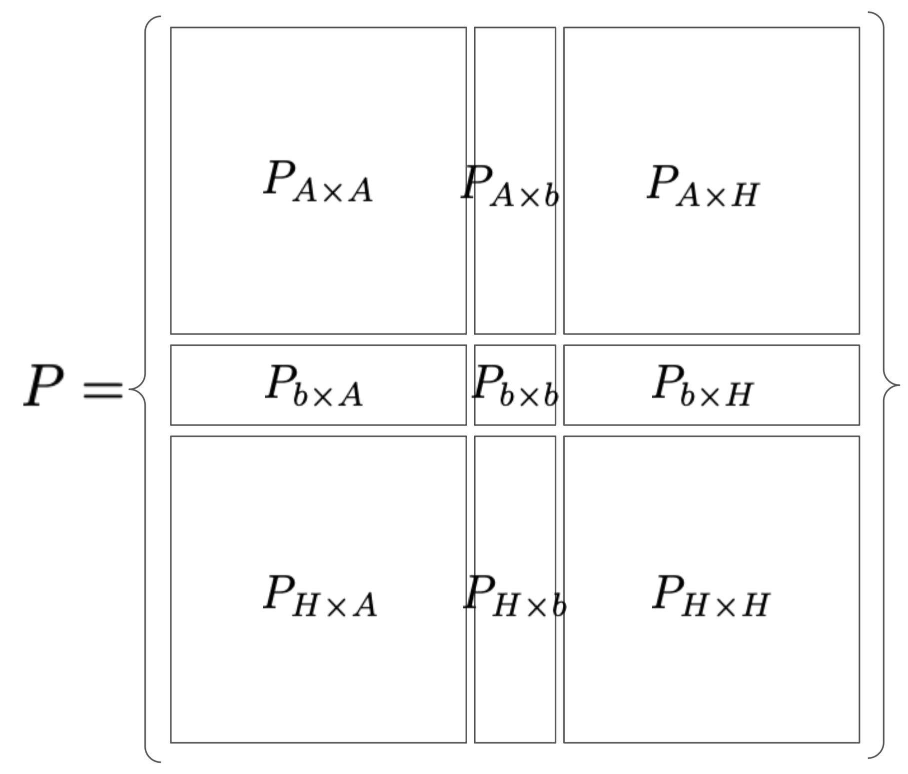


Each equation is encoded as a token $e_i = (a_i, b_i, h_i)$,where $H = (h_0, h_1, \ldots h_N)$ is an optional embedding matrix (either learned or predefined).

We append a query token $e_{N+1} = (x_0, 1_{1+K})$ to the sequence, where x_0 represents test data.

Re-parameterization of weight matrix

Consider the following block re-parametrization of weight matrix P (same for matrix Q):

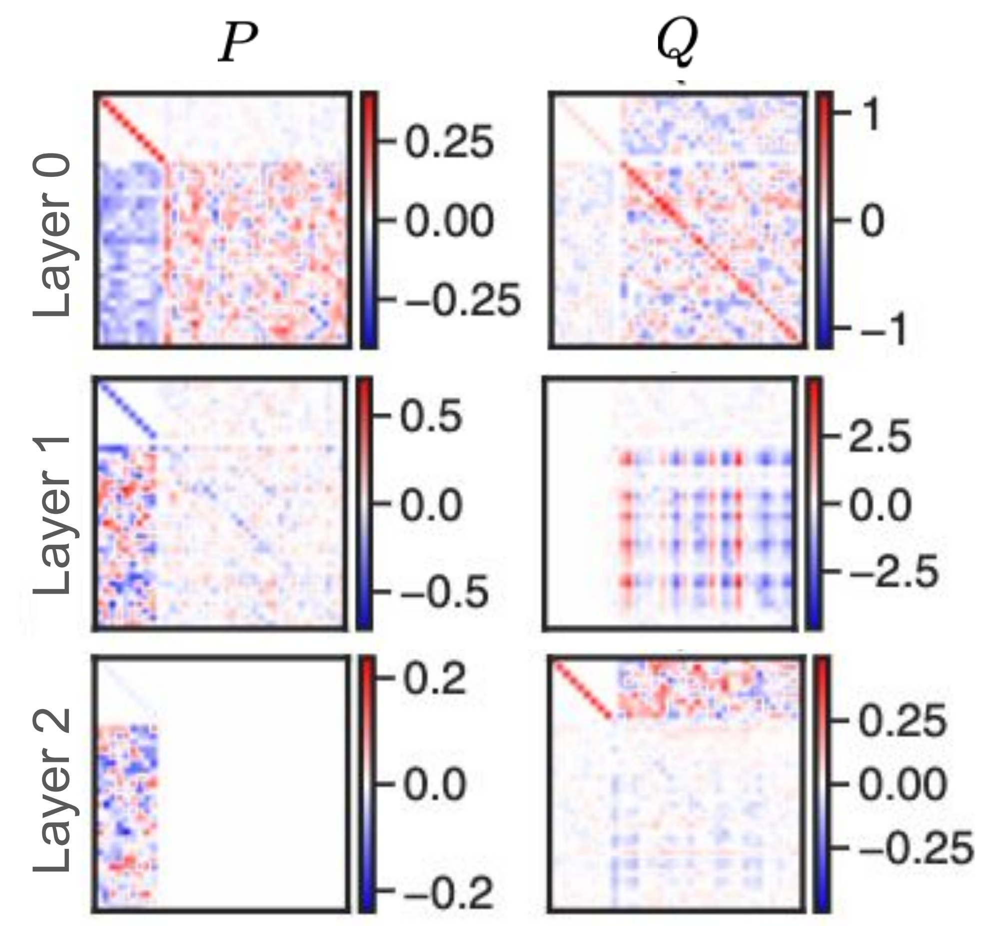


- Square matrices represented as scalar times identity, e.g. $P_{A\times A} = p_{A\times A} I$.
- Rectangular matrices represented as scalar times a vector of ones, e.g. $P_{A\times b} = p_{A\times b} 1$

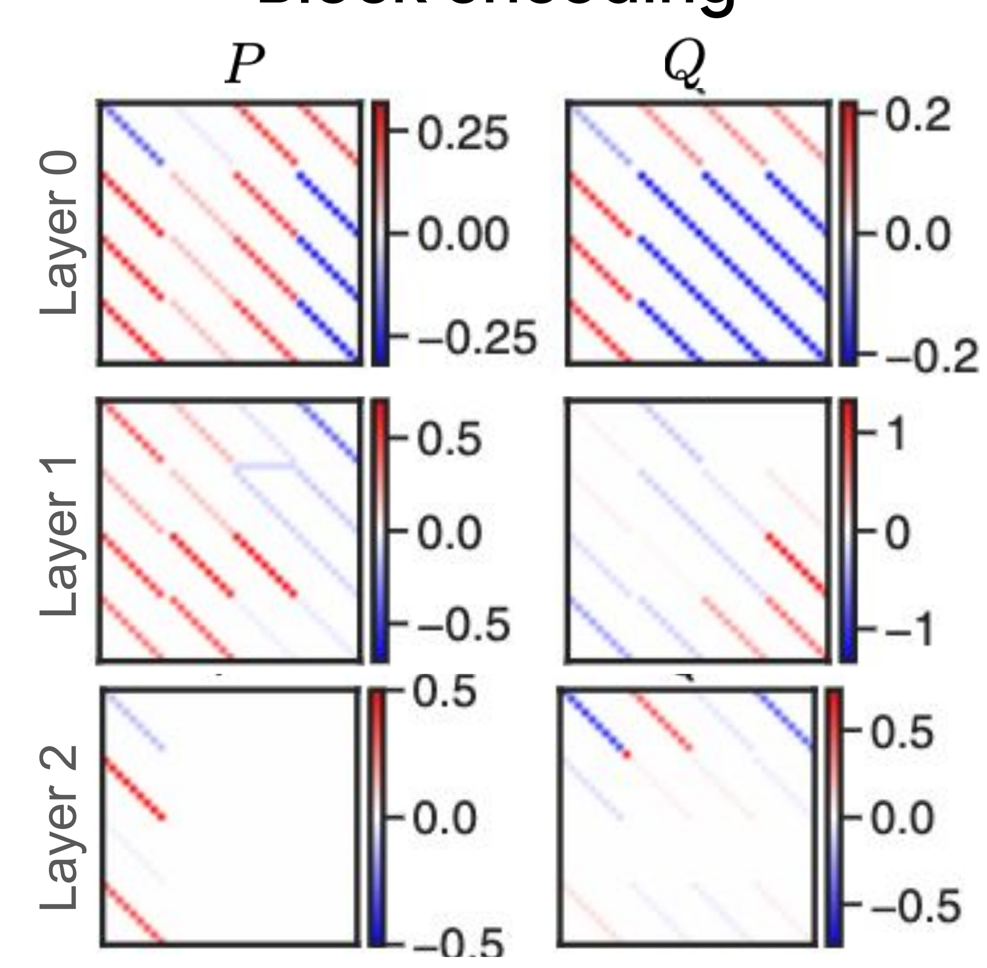
Motivation:

- Elements of A are sampled independently, => there is no bias for any dimension.
- **Efficiency:** identity matrices speed up computation
- **Performance:** comparable loss to training with full matrices.
- **Generalization:** Decouples problem dimension (N) from model parameters, enabling application and fine-tuning to different input sizes.

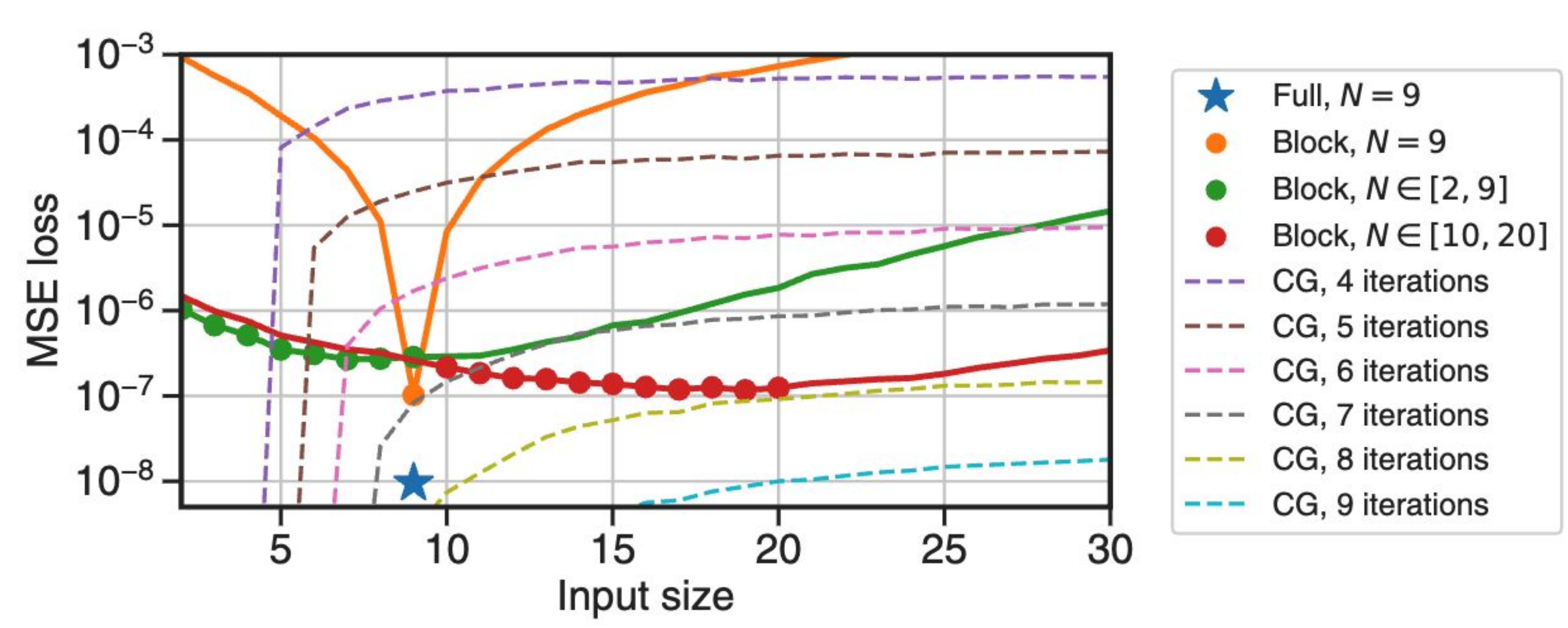
Full encoding



Block encoding



Experiments



- **Full, *N=9*.** Full encoding, trained on *N=9* problems only + 27-dim learned embedding *H*. This model cannot generalize to matrices of other sizes, but it achieves the best performance for problems of size *N=9*.
- **Block, *N=9*.** Block encoding, trained on *N=9* problems only + three *NxN* fixed identity matrices H. The generalization quality is limited.
- **Block, *N* ∈ *[2,9]*.** Block encoding, trained on sizes *N* ∈ *[2,9]* + three *NxN* fixed identity matrices H. Generalizes well beyond its training sizes.
- **Block, *N* ∈ *[10,20]*.** Block encoding, fine-tuned from model *N* ∈ *[2,9]* above on sizes *N* ∈ *[10,20]*. Generalizes well beyond its training sizes.